Successive, radial, and parabolic limits

Successive or Iterated Limits

This resource involves making the variables tend one at a time in a successive form. These limits are not limits along paths that pass through the point.

$$L_{1,2} = \lim_{y \to y_0} \left(\lim_{x \to x_0} f(x,y) \right) = \lim_{y \to y_0} g(y) \quad \text{if} \quad g(y) \text{ exists}$$

$$L_{2,1} = \lim_{x \to x_0} \left(\lim_{y \to y_0} f(x,y) \right) = \lim_{x \to x_0} h(x) \quad \text{if} \quad h(x) \text{ exists}$$

If the successive limits exist, are finite and distinct, I can guarantee the non-existence of the Double Limit. If they are equal, I cannot determine the existence of the double limit. If one of the successive does not exist, I cannot conclude about the existence or non-existence of the double limit.

Radial Limit

We approach to (x_0, y_0) along lines with the equation:

$$y - y_0 = m(x - x_0)$$

$$x - x_0 = m(y - y_0)$$

$$\lim_{\substack{x \to x_0 \\ y \to m(x-x_0)+y_0}} f(x,y) \quad \text{or} \quad \lim_{\substack{y \to y_0 \\ x \to m(y-y_0)+x_0}} f(x,y)$$

Performing the limit in either of the two forms is indifferent. If the radial limit depends on m, I can assure the non-existence of the double limit. If it does not depend on m, I cannot assure the existence of the double limit.

Parabolic Limit

We approach to (x_0, y_0) along parabolas that pass through the point, with the equation:

$$y - y_0 = a(x - x_0)^2$$

$$x - x_0 = b(y - y_0)^2$$

$$\lim_{\substack{x \to x_0 \\ y \to m(x-x_0)+y_0}} f(x,y) \quad \text{or} \quad \lim_{\substack{y \to y_0 \\ x \to m(y-y_0)+x_0}} f(x,y)$$

Performing the limit in either of the two forms is indifferent. If the radial limit depends on m, I can assure the non-existence of the double limit. If it does not depend on m, I cannot assure the existence of the double limit.

Conclusion

Let $z = f(x, y)$ be a function, and (x_0, y_0) a point. I am interested in the behavior of the function in a neighborhood of the point (as close as I wish).

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L, \quad L \in \mathbb{R}$$

If the double limit exists we call it L, with L being a number that belongs to the real numbers, then: L is equal to the Radial, Parabolic, and Successive (Iterated) limits.

$$L = L_r = L_p = L_{1_2} = L_{2_1}$$

It may happen that L exists, but L_{x_2} or L_{y_1}; or that L exists but neither L_{1_2} nor L_{2_1} exist.

If the double limit exists, then the radial limits and the parabolic limits exist, and all limits along any path passing through the point are equal to the double limit. Furthermore, if both successive limits exist and are equal, they are equal to the double limit. If we find two paths passing through the point along which the limits are different, we can guarantee the non-existence of the double limit. If the successive limits exist and are distinct, we can guarantee the non-existence of the double limit."